

Spectral gap properties and convergence to stable laws for affine random walks on \mathbb{R}^d

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Abstract

We consider a Markov chain $\{X_n\}_{n=1}^\infty$ on \mathbb{R}^d defined by the stochastic recursion $X_n = M_n X_{n-1} + Q_n$ where (Q_n, M_n) are i.i.d. random variables which take values in the affine group of the vector space \mathbb{R}^d . Under natural hypothesis on (Q_n, M_n) , including negativity of the dominant Lyapunov exponent of the product of the matrices M_n , the transition operator P of the chain has a unique stationary measure η and it is known that η is α -homogeneous at infinity for some $\alpha > 0$ depending on the law of M_n . We show spectral gap properties for P and for a class of Fourier operators $P_v (v \in \mathbb{R}^d)$, on function spaces on \mathbb{R}^d of Hölder type. If $d > 1$, we consider the Birkhoff sums $S_n = \sum_{k=0}^n X_k$ and we show that the normalized sums converge to a normal law ($\alpha \geq 2$) or to an α -stable law ($\alpha < 2$) on \mathbb{R}^d , and these laws are fully non-degenerate. For $d = 1$ such results were first obtained in [21]. Here we describe their natural extension to the general multidimensional setting. The corresponding analysis of the characteristic function of S_n is based on the spectral gap properties of P_v and on the homogeneity at infinity of η and of a family of companion measures η_v .

1 Introduction and main results

We consider the vector space $V = \mathbb{R}^d$ endowed with the scalar product $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$ and the norm $|x| = \left(\sum_{i=1}^d |x_i|^2 \right)^{1/2}$. We denote by $H = V \rtimes G$ the affine group of V , with $G = GL(d, \mathbb{R})$, i.e. the set of maps h of the form $hx = gx + b (b \in V, g \in G)$. Let μ be a probability measure on H and $x \in V$. We denote by \mathbb{P} the product measure $\mu^{\otimes \mathbb{N}}$ on $\Omega = H^{\mathbb{N}}$ and we consider the recurrence relation with random coefficients:

$$X_0^x = x, \quad X_n^x = M_n X_{n-1}^x + Q_n \quad (n \geq 1), \quad (1.1)$$

where $(Q_n, M_n) \in H$ are i.i.d. random variables with generic copy (Q, M) and with law μ . Let $\bar{\mu}$ be the projection of μ on G , i.e. the law of M_n , and let $[\text{supp} \bar{\mu}]$ be the closed sub-semigroup generated by the support of $\bar{\mu}$. We will denote by P the corresponding Markov operator on $C_b(V)$, the space of continuous bounded functions on V :

$$P\varphi(x) = \int \varphi(gx + b) d\mu(h), \quad \varphi \in C_b(V).$$

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Our hypothesis will imply that the above recursion (1.1) has a unique stationary measure η which satisfies $P\eta = \eta$ and has an unbounded support. The probability measure η is the limit distribution of X_n^x . A remarkable property of η is its "homogeneity at infinity", a property which was first observed in [31] for the tails of η , extended to the general case in [34] and further developed in [2, 6, 13], under special conditions. See [17] for a survey of [34] as well for a precise description of the homogeneity property of η , proved in [22].

In this paper we are interested in the limit behavior of the sum $S_n^x = \sum_{k=0}^n X_k^x$, conveniently normalized. For $d = 1$ this question is closely related to the slow diffusion behavior of a simple random walk on \mathbb{Z} in a random medium (See [33, 39]). The similar problem for a finitely supported random walk on \mathbb{Z} in a random medium is connected to the study of a recurrence relation of the form (1.1) (See [14, 29]).

For $d = 1$, and under aperiodicity conditions, the limit behavior of S_n^x is described in [21]. For $d \geq 1$, in the case where M_n takes values in the similarity group of V , the limit behavior of S_n^x is described in [5]; the homogeneity at infinity result of [6] plays an essential role in the proof. Here we consider the case where $[\text{supp}\bar{\mu}]$ is "large", a case which is opposite to the case of [5], and we will need detailed information on the stationary law η of P . Also as in [21, 5], a basic role will be played by the spectral properties of the operators P_v ($v \in \mathbb{R}$) defined by $P_v\varphi = P(\mathcal{X}_v\varphi)$, where $\mathcal{X}_v(x) = e^{i\langle v, x \rangle}$. Furthermore, the homogeneity at infinity of η implies that the dominant eigenvalue of P_v has an asymptotic expansion at 0 in term of fractional powers of $|v|$. These properties allow us to develop a detailed analysis and to prove limit theorems. More generally, it turns out that, in the context of random walks associated with proximal semigroup actions, spectral gap properties are valid in certain functional spaces for large classes of random walks. These properties are valid without density hypothesis on μ or $\bar{\mu}$. See [7, 8, 10, 15, 11, 16, 19] for different classes of situations where analogous ideas are used. Here V can be considered as a boundary (see [12]) for the random walk defined by μ , and we will use spectral gap properties for P_v ($v \in V$) in functional spaces of Hölder type. In [8] and [11] the relevant spaces are L^2 -spaces, while in [7, 10, 16], they are of mixed type. This type of analysis is not restricted to homogeneous spaces of Lie groups as shown in [37] for certain classes of Lipschitz maps instead of affine maps. We will follow as closely as possible the notations in [5], but the geometrical situation brings out important new aspects with respect to [5] and [6]. When convenient we will take advantage of the detailed calculations already developed in [5]. The arguments of the proofs will be closely related to those in [21, 5]; but we will develop them from scratch in the more general framework of our paper, since the results of [20, 22] play an essential role.

The asymptotics of products of random matrices (See [18, 3, 23]) will play an important role here, and we need to give corresponding notations. We say that a semigroup $\Gamma \subset G$ is *strongly irreducible* if no finite union of proper subspaces of V is Γ -invariant. Also we say that $g \in G$ is *proximal* if g has a dominant eigenvalue $\lambda(g) \in \mathbb{R}$ which is the unique eigenvalue of g such that $|\lambda(g)| = \lim_{n \rightarrow \infty} |g^n|^{1/n}$ where $|g| = \sup\{|gx| : |x| = 1\}$. We say that Γ satisfies condition *i-p* if Γ is strongly irreducible and contains a proximal element γ . It is proved in [15, 20] that condition *i-p* for Γ and its Zariski closure $Zc(\Gamma)$ are equivalent. Since $Zc(\Gamma)$ is a closed Lie subgroup of G with a finite number of connected components, condition *i-p* can be checked in examples (see Section 5 for some examples). Under this condition, the limit set $L(\Gamma) \subset \mathbb{P}^{d-1}$ is the unique Γ -minimal subset of the projective space \mathbb{P}^{d-1} and $L(\Gamma)$ is the closure of the set of attracting fixed points of the proximal elements in Γ .

For $s \geq 0$, we denote

$$\begin{aligned} \kappa(s) &= \lim_{n \rightarrow \infty} (\mathbb{E}|M_n \cdots M_1|^s)^{1/n}, \\ s_\infty &= \sup\{s \geq 0; \kappa(s) < \infty\}. \end{aligned}$$

If $\mathbb{E}(\log^+|M|) < +\infty$, we know that the Lyapunov exponent

$$L(\bar{\mu}) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(\log|M_n \cdots M_1|)$$

is well defined, $L(\bar{\mu}) = \kappa'(0_+)$ if $s_\infty > 0$. If condition *i-p* is satisfied and $s_\infty > 0$, then $\log \kappa(s)$ is strictly convex on $[0, \infty)$, hence if $\lim_{s \rightarrow s_\infty} \kappa(s) \geq 1$, there exists a unique $\alpha \in]0, \infty[$ with $\kappa(\alpha) = 1$.

Our hypothesis here is the following condition *C* (See [17]):

- C_0 $[\text{supp} \bar{\mu}]$ satisfies condition *i-p*,
- C_1 $\mathbb{E}(\log^+|M| + \log^+|Q|) < \infty$,
- C_2 $L(\bar{\mu}) < 0$, $s_\infty > 0$, $\lim_{s \rightarrow s_\infty} \kappa(s) \geq 1$,
- C_3 $\mathbb{E}(|M|^\alpha \log^+|M| + |Q|^\alpha) < \infty$,
- C_4 $\text{supp} \mu$ has no fixed point in V .

A real number $t \in \mathbb{R}$ defines a dilation on V which is denoted by $v \rightarrow t.v$, and we extend this notation to the action of \mathbb{R} on measures.

Let \bar{P} be the Markov operator on V defined by

$$\bar{P}\varphi(v) = \int \varphi(gv) d\bar{\mu}(g), \quad \text{if } \varphi \in C_b(V).$$

We observe that \bar{P} can be interpreted as the linearisation of P at infinity. We denote by ℓ^s the homogeneous measure on \mathbb{R}_+^* defined by $\ell^s(dt) = \frac{dt}{t^{s+1}}$. It is proved in [22] that if $d > 1$ and condition *C* is valid, there exists $c > 0$ and a probability measure σ_α on the unit sphere \mathbb{S}^{d-1} such that the following weak convergence is valid on $V \setminus \{0\}$:

$$\lim_{t \rightarrow 0_+} t^{-\alpha}(t.\eta) = c\sigma_\alpha \otimes \ell^\alpha = \Lambda. \quad (1.2)$$

Here Λ is defined by the above convergence and satisfies $t.\Lambda = t^\alpha \Lambda$ for $t > 0$, and we have $\bar{P}\Lambda = \Lambda$. We observe that the equation $\bar{P}\Lambda = \Lambda$ is a limiting form of the stationarity equation $P\eta = \eta$. The proof is based on the general renewal theorem of [32] and on the spectral gap property of the operator on the projective space defined by twisted convolution with $\bar{\mu}$ (See [20]).

More generally, if η is a probability measure such that the above convergence (1.2) is valid, we will say that η is α -homogeneous at infinity. If a Radon measure Λ satisfies $t.\Lambda = t^\alpha \Lambda$ for $t > 0$, we will say that Λ is α -homogeneous. A probability η on V is said to be stable if for every integer n there exists a similarity h_n of the form $h_n(x) = a_n x + b_n$ ($a_n > 0, b_n \in V$) such that the n^{th} convolution power of η is the push forward of η by h_n . If $a_n = n^{1/\alpha}$, we say that η is α -stable.

If $\text{supp} \bar{\mu}$ has no invariant convex cone in V , then Λ is symmetric and $\sigma_\alpha \otimes \ell^\alpha$ is the unique Radon measure defined by the following conditions:

$$\sigma_\alpha \text{ is a probability on } \mathbb{S}^{d-1}, \quad \bar{P}(\sigma_\alpha \otimes \ell^\alpha) = \sigma_\alpha \otimes \ell^\alpha, \quad t.(\sigma_\alpha \otimes \ell^\alpha) = t^\alpha(\sigma_\alpha \otimes \ell^\alpha), \quad \text{for all } t > 0.$$

See [22] for more detail. In Section 5 below we give information on σ_α and examples of the typical situations which can occur. In any case the projection on \mathbb{P}^{d-1} of $\text{supp} \sigma_\alpha$ is equal to the limit set $L([\text{supp} \bar{\mu}])$ in \mathbb{P}^{d-1} .

We will write g^* for the transposed map of $g \in G$, $\bar{\mu}^*$ for the push-forward of $\bar{\mu}$ by $g \rightarrow g^*$. Also for $x \in V$, we write x^* for the linear form $x^*(y) = \langle x, y \rangle$. The exponential $e^{i\langle x, y \rangle}$ will be denoted by $\mathcal{X}_x(y)$ and the characteristic function of a probability measure π on V will be defined by

$$\hat{\pi}(x) = \int_V \mathcal{X}_x(y) d\pi(y).$$

Coming back to the affine situation, we will write

$$m = \int x d\eta(x), \quad m_\alpha = \kappa'(\alpha_-).$$

The calculation of the limit law of S_n^x will involve considering the companion recursion :

$$W_0 = 0, \quad W_n = M_n^*(W_{n-1} + v), \quad (1.3)$$

where $v \in V$ is a fixed vector. We will denote by T_v the corresponding transition operator, i.e.

$$T_v(\varphi) = \int \varphi(g^*(x + v)) d\bar{\mu}(g).$$

Then as above, the unique stationary measure η_v of T_v satisfies the weak convergence on $V \setminus \{0\}$:

$$\lim_{t \rightarrow 0_+} t^{-\alpha}(t.\eta_v) = \Delta_v \neq 0, \quad (1.4)$$

and η_v, Δ_v satisfy

$$\eta_{tv} = t.\eta_v, \quad \Delta_{tv} = t.\Delta_v \quad \text{for } t \in \mathbb{R}^*, \quad \bar{P}^* \Delta_v = \Delta_v, \quad \Delta_{tv} = t^\alpha \Delta_v \text{ for } t > 0,$$

where, as above, \bar{P}^* is associated with $\bar{\mu}^*$.

In order to state our first main result, we need to define a kind of Fourier transform $\tilde{\Lambda}$ of Λ . If $\alpha \in [0, 2]$, we define $\tilde{\Lambda}$ as follows:

$$\begin{aligned} \tilde{\Lambda}(y) &= \int (\mathcal{X}_y(x) - 1) d\Lambda(x), \quad \text{if } 0 < \alpha < 1, \\ \tilde{\Lambda}(y) &= \int \left(\mathcal{X}_y(x) - 1 - i \frac{\langle x, y \rangle}{1 + |\langle x, y \rangle|^2} \right) d\Lambda(x), \quad \text{if } \alpha = 1, \\ \tilde{\Lambda}(y) &= \int (\mathcal{X}_y(x) - 1 - i \langle x, y \rangle) d\Lambda(x), \quad \text{if } 1 < \alpha < 2, \\ \tilde{\Lambda}(y) &= -\frac{1}{4} \int \langle y, x \rangle^2 d\sigma_2(x), \quad \text{if } \alpha = 2. \end{aligned}$$

The function $\tilde{\Lambda}$ satisfies

$$\tilde{\Lambda}(ty) = t^\alpha \tilde{\Lambda}(y) \quad \text{for } t > 0, \quad \bar{P}^* \tilde{\Lambda} = \tilde{\Lambda}, \quad \text{and} \quad \operatorname{Re} \tilde{\Lambda}(y) < 0 \text{ for } y \neq 0.$$

We will use also the function $\tilde{\Lambda}^1$ defined by $\tilde{\Lambda}^1(y) = \tilde{\Lambda}(\bar{y}) \mathbf{1}_{[1, \infty[}(|y|)$, where \bar{y} denotes the projection of $y \in V \setminus \{0\}$ on \mathbb{S}^{d-1} .

The Fourier transform of the limit law of S_n^x for $\alpha \in [0, 2]$ will be shown to be equal to $e^{C_\alpha(v)} = \Phi_\alpha(v)$ where the function $C_\alpha(v)$ is defined by

$$C_\alpha(v) = \begin{cases} \alpha m_\alpha \Delta_v(\tilde{\Lambda}^1), & \text{if } \alpha \in (0, 1) \cup (1, 2]; \\ m_1 \Delta_v(\tilde{\Lambda}^1) + i\gamma(v), & \text{if } \alpha = 1, \end{cases} \quad (1.5)$$

with

$$\gamma(v) = \iint \left[\frac{\langle y + v, x \rangle}{1 + |\langle y + v, x \rangle|^2} - \frac{\langle v, x \rangle}{1 + |x|^2} - \frac{\langle y, x \rangle}{1 + |\langle y, x \rangle|^2} \right] d\Lambda(x) d\eta_v(y). \quad (1.6)$$

(See the proof of Proposition 2.7.) We have that for $t > 0$

$$C_\alpha(tv) = t^\alpha C_\alpha(v) \quad \text{if } \alpha \neq 1, \quad \text{and} \quad C_1(tv) = tC_1(v) + i \langle v, \beta(t) \rangle,$$

where $\beta(t) = \int \left(\frac{tx}{1 + |tx|^2} - \frac{tx}{1 + |x|^2} \right) d\Lambda(x)$. Hence $e^{C_\alpha(v)}$ is the Fourier transform of an infinitely divisible probability measure which belongs to an α -stable convolution semigroup (see [26, 28, 38]).

If $\alpha > 2$, the following covariance form q of η will enter in the formulas below,

$$q(x, y) = \int \langle x, \xi - m \rangle \langle y, \xi - m \rangle d\eta(\xi).$$

We will write $z = \mathbb{E}(M)$ for the averaged operator of M if $\alpha > 1$. One sees easily that the operator $\mathbb{E}M$ exists and has spectral radius less than $\kappa(\alpha) = 1$, hence in particular $I - z^*$ is invertible.

We have the following limit theorem for the partial sums S_n^x .

Theorem 1.1. Assume that the probability measure μ on $H = V \rtimes G$ satisfies condition C above. Then if $\dim V > 1$, we have for any $x \in V$,

1) If $\alpha > 2$, $\frac{1}{\sqrt{n}}(S_n^x - nm)$ converges in law to the normal law on V with the Fourier transform

$$\Phi_{2+}(v) = \exp(-q(v, v)/2 - q(v, (I - z^*)^{-1}z^*v)).$$

2) If $\alpha \in (0, 2)$, let $t_n = n^{-1/\alpha}$ and

$$d_n = \begin{cases} 0, & \alpha \in (0, 1); \\ n\delta(t_n), & \alpha = 1; \\ nt_n, & \alpha \in (1, 2), \end{cases}$$

with $\delta(t) = \int_V \frac{tx}{1 + |tx|^2} d\eta(x)$ for $t > 0$. Then $(t_n S_n^x - d_n)$ converges in law to the α -stable law with the Fourier transform $\Phi_\alpha(v) = \exp(C_\alpha(v))$, with $C_\alpha(v)$ given above.

Furthermore if $\alpha = 1$, then for some constant $K_\dagger > 0$,

$$|\delta(t)| \leq \begin{cases} K_\dagger |t| |\log |t||, & \text{for } |t| \leq \frac{1}{2}; \\ K_\dagger |t|, & \text{for } |t| > \frac{1}{2}. \end{cases}$$

3) If $\alpha = 2$, then $\frac{1}{\sqrt{n \log n}}(S_n^x - nm)$ converges in law to the normal law with Fourier transform

$$\Phi_2(v) = \exp(C_2(v)), \quad \text{where} \quad C_2(v) = -\frac{1}{4} \int (\langle v, w \rangle)^2 + 2 \langle v, w \rangle \eta_v(w^*) d\sigma_2(w).$$

4) In all cases, the limit laws are fully non degenerate.

The proof of Theorem 1.1 is based on the method of characteristic functions. The Fourier transform of S_n^x can be expressed in terms of iterates of the Fourier operator P_v defined above. This operator acts as a bounded operator on a certain Banach space $\mathbb{B}_{\theta, \varepsilon, \lambda}$ (defined below) of unbounded functions on V and has “nice” spectral properties on $\mathbb{B}_{\theta, \varepsilon, \lambda}$. Moreover $P_0 = P$ and the spectral properties of P_v allow to control the perturbation P_v of P as well as its dominant eigenvalue $k(v)$. Theorem 1.1 follows from the asymptotic expansion of $k(v)$ at $v = 0$. The spectral properties of P_v follow from a theorem of Ionescu-Tulcea and Marinescu based on certain functional inequalities proved below.

We denote by $r(U)$ the spectral radius of a bounded linear operator U . The spectral properties of P_v are described by the:

Theorem 1.2. If $v \in V$, the operator P_v on $\mathbb{B}_{\theta, \varepsilon, \lambda}$ defined by $P_v f = P(\mathcal{X}_v f)$ has the following properties:

- 1) P_v is a bounded operator with spectral radius at most 1,
- 2) If $v \neq 0$, $r(P_v) < 1$,
- 3) If $v = 0$ and π_0 is the projection on $\mathbb{C}\mathbf{1}$ defined by $\pi_0 \varphi = \eta(\varphi)\mathbf{1}$, we have for any $\varphi \in \mathbb{B}_{\theta, \varepsilon, \lambda}$:

$$P_0 \varphi = \pi_0 \varphi + Q \varphi$$

where $Q\pi_0 = \pi_0 Q = 0$ and $r(Q) < 1$.

4) If v is small, P_v has a unique eigenvalue $k(v)$ with $|k(v)| = r(P_v)$. Furthermore there exists a one dimensional projection π_v and a bounded operator Q_v such that $Q_v \pi_v = \pi_v Q_v = 0$, $r(Q_v) < |k(v)|$ and

$$P_v \varphi = k(v) \pi_v \varphi + Q_v \varphi, \quad \text{for any } \varphi \in \mathbb{B}_{\theta, \varepsilon, \lambda}.$$

Furthermore $k(v), \pi_v, Q_v$ depend continuously on v .

These spectral properties will allow us to reduce the study of the iterated operator P_v^n to the study of its dominant eigenvalue $k^n(v)$; hence $k(v)$ plays here the role of a characteristic function for the convolution operator P defined by μ on $C_b(V)$.

The asymptotic behavior of $k(v)$ at $v = 0$ is given by the

Theorem 1.3. *Let $v \in V \setminus \{0\}$ and let $C_\alpha(v)$ be given by (1.5).*

1) *If $0 < \alpha < 1$, then*

$$\lim_{t \rightarrow 0_+} \frac{k(tv) - 1}{t^\alpha} = C_\alpha(v).$$

2) *If $\alpha = 1$, then*

$$\lim_{t \rightarrow 0_+} \frac{k(tv) - 1 - i \langle v, \delta(t) \rangle}{t} = C_1(v).$$

3) *If $1 < \alpha < 2$, then*

$$\lim_{t \rightarrow 0_+} \frac{k(tv) - 1 - i \langle v, tm \rangle}{t^\alpha} = C_\alpha(v).$$

4) *If $\alpha = 2$, then*

$$\lim_{t \rightarrow 0} \frac{k(tv) - 1 - i \langle v, tm \rangle}{t^2 |\log |t||} = 2C_2(v).$$

5) *If $\alpha > 2$, then*

$$\lim_{t \rightarrow 0} \frac{k(tv) - 1 - i \langle v, tm \rangle}{t^2} = C_{2+}(v),$$

with

$$C_{2+}(v) = -\frac{1}{2}q(v, v) - q(v, (I - z^*)^{-1}z^*v).$$

As in [21] and [5], the proof of Theorem 1.3 is based on an intertwining relation between the families of operators P_v and T_v and on the homogeneity at infinity of η , η_v proved in [22]; this relation allows us to express $k(v)$ in terms of the stationary measure η and an eigenfunctional for T_v .

Remark 1.4.

a) We may observe that, if we add stronger moment conditions (of order greater than 4), part 1 of Theorem 1.1, i.e. convergence to a normal law, follows from the main result of [25], which is valid also for more general Lipschitz maps of V into itself.

b) For $\alpha \in [0, 2]$, the limit law of S_n^x is a multidimensional α -stable law (see e.g. [26, 28, 36]) where α -stability holds with respect to the action of the dilation group \mathbb{R}_+^* . In particular the limit law is infinitely divisible and belongs to a convolution semigroup of \mathbb{R}^d . This remarkable fact follows from the homogeneity of Δ_v with respect to v , hence of the formula for $C_\alpha(v)$. In Section 5 we will give more information on the function C_α which satisfies in particular $\operatorname{Re} C_\alpha(v) < 0$ for $v \neq 0$.

c) The fact that the stability group here is \mathbb{R}_+^* , instead of a more complex one as in [5], is a consequence of the following property (see [23, 24]): the closed semigroup of \mathbb{R}_+^* generated by the modulus of the dominant eigenvalues for the proximal elements in $[\operatorname{supp} \bar{\mu}]$ is equal to \mathbb{R}_+^* . This can be compared with the situation of [5] where semi-stable laws in the sense of [36, p.204] appear as limits.

d) Since P_v satisfies $r(P_v) < 1$ for any $v \neq 0$, we can prove also local limit theorems for the sums S_n^x (see [5] for the case where $[\operatorname{supp} \bar{\mu}]$ consists of similarities).

After writing this paper, we became aware of the reference [9] in which closely related limit theorems for S_n^x were obtained under stronger hypothesis than here; in particular a density hypothesis on μ is assumed in [9] and $\alpha = 1$ or 2 is excluded.

2 Homogeneity at infinity of μ -stationary measures

The following gives the existence and elementary properties of the stationary law of X_n^x in our context.

Proposition 2.1. *Assume that μ satisfies condition C. Let*

$$R_n = Q_1 + \sum_{k=1}^{n-1} M_1 \cdots M_k Q_{k+1}.$$

Then R_n converges a.e. to

$$R = Q_1 + \sum_{k=1}^{\infty} M_1 \cdots M_k Q_{k+1}$$

and the law of X_n^x converges to the law η of R . Furthermore, η has no atom, gives measure zero to every affine subspace and $\mathbb{E}(|R|^\theta) = \int |x|^\theta d\eta(x) < \infty$ if $\theta < \alpha$.

Remark 2.2. One can show that, except for the last assertion, the above statement is also valid under conditions C_0 , C_1 , C_4 and $L(\bar{\mu}) < 0$.

Proof. The convergence proofs are based on known arguments (see [3, 4]), hence we give only a sketch. In our setting, if $s < \alpha$, we have by definitions of $\kappa(s)$:

$$\mathbb{E}(|M_1 \cdots M_k|^s) = \mathbb{E}(|M_k \cdots M_1|^s) \leq C(\kappa(s) + \epsilon)^k$$

for some $C > 0$ and $0 < \epsilon < \kappa(\alpha) - \kappa(s)$. Also $\mathbb{E}(|Q_k|^s) = \mathbb{E}(|Q_1|^s) \leq \mathbb{E}(|Q_1|^\alpha)^{s/\alpha} < \infty$. It follows if $m > n$,

$$\mathbb{E}(|R_m - R_n|^s) \leq C(\mathbb{E}(|Q_1|^\alpha))^{s/\alpha} \sum_{k=n}^{m-1} (\kappa(s) + \epsilon)^k < \infty.$$

Hence $\lim_{m,n \rightarrow \infty} \mathbb{E}(|R_m - R_n|^s) = 0$. The convergence a.e. of R_n to R follows. The same calculation shows $\mathbb{E}(|R|^\theta) < \infty$ if $\alpha \leq 1$ and $\theta < \alpha$. If $\alpha > 1$ and $\theta \in [1, \alpha]$, we use Minkowski inequality in $\mathbb{L}^\theta(\Omega)$ and the independence of $M_1 \cdots M_{k-1}, Q_k$ to get that :

$$\mathbb{E}(|R|^\theta) \leq C \mathbb{E}(|Q_1|^\theta) \left[\sum_{k=1}^{\infty} (\kappa(\theta) + \epsilon)^{k/\theta} \right]^\theta < \infty,$$

if ϵ satisfies $\kappa(\theta) + \epsilon < 1$.

The fact that η has no atom is proved as follows.

Let $A \subset V$ be the set of atoms of η . Then A is countable and $\sum_{x \in A} \eta(\{x\}) \leq 1$. It follows that, for every $\epsilon > 0$, the set $\{x \in A; \eta(x) \geq \epsilon\}$ is finite; in particular, $\sup_{x \in A} \eta(\{x\}) = c$ is attained. Let $A_0 = \{x \in A; \eta(x) = c\}$. Since $P\eta = \eta$, we have $hA_0 = A_0$ if $h \in \text{supp}\mu$. Then the barycenter of A_0 is a $\text{supp}\mu$ -invariant point, which is excluded by condition C_4 .

Assume now that there exists an affine subspace W of positive dimension such that $\eta(W) > 0$, and let \mathcal{W} be the set of affine subspaces of minimum dimension r with $\eta(W) > 0$. If $r = 0$, the contradiction follows from above. If $r > 0$, we observe that for any $W, W' \in \mathcal{W}$ with $W \neq W'$, we have $\eta(W \cap W') = 0$ since $\dim(W \cap W') < \dim W$. Then as above $\sup_{W \in \mathcal{W}} \eta(W) = c'$ is attained. If $\mathcal{W}_0 = \{W \in \mathcal{W} : \eta(W) = c'\}$, we have $h\mathcal{W}_0 = \mathcal{W}_0$ for any $h \in \text{supp}\mu$. Let Γ be the closed subgroup of H generated by $\text{supp}\mu$, hence $h\mathcal{W}_0 = \mathcal{W}_0$ for any $h \in \Gamma$. Then the subset Γ_0 of Γ , which leaves invariant any $W \in \mathcal{W}_0$, is a finite index subgroup of Γ . Since $L(\bar{\mu}) < 0$, $[\text{supp}\mu]$ has a element g with $|g| < 1$. Assume $h \in [\text{supp}\mu]$ has linear part g and observe that h has a unique

fixed point $x \in V$ which is attracting. Since Γ_0 has finite index in Γ , we can find $p \in \mathbb{N}$ such that $h^p \in \Gamma_0$. Then for any $y \in W$ with $W \in \mathcal{W}_0$, we have

$$\lim_{n \rightarrow \infty} h^{pn} y = x.$$

Since $h^{pn} y \in W$, we get $x \in W$, hence

$$x \in \bigcap_{W \in \mathcal{W}_0} W \neq \emptyset.$$

It follows that Γ leaves invariant the nontrivial affine subspace $\bigcap_{W \in \mathcal{W}_0} W$. If $\dim \bigcap_{W \in \mathcal{W}_0} W = 0$, we have constructed a point invariant under Γ , which contradicts conditions C_4 . If $\dim \bigcap_{W \in \mathcal{W}_0} W > 0$, the direction of this affine subspace is a proper $\text{supp}\bar{\mu}$ -invariant linear subspace, which contradicts condition i - p for $\text{supp}\bar{\mu}$. \square

For $\kappa(s)$ we have the following (see [20]):

Proposition 2.3. *Assume $[\text{supp}\bar{\mu}]$ satisfies conditions i - p . Then $\log \kappa(s)$ is strictly convex on $[0, s_\infty[$. If $s_\infty = \infty$, we have:*

$$\begin{aligned} \lim_{s \rightarrow \infty} \frac{\log \kappa(s)}{s} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sup \{ \log |g| : g \in [\text{supp}\bar{\mu}]^n \} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sup \{ \log r(g) : g \in [\text{supp}\bar{\mu}]^n \} \end{aligned}$$

In particular, the condition $\kappa(s) < 1$ on $]0, \infty[$ is equivalent to $r(g) \leq 1$ on $[\text{supp}\bar{\mu}]$, and if $\lim_{s \rightarrow s_\infty} \kappa(s) \geq 1$ there exists a unique $\alpha \in]0, s_\infty]$ such that $\kappa(\alpha) = 1$.

Remark 2.4. Regularity properties of $\kappa(s)$, not used here, are proved in [20]. In particular, $\kappa(s)$ is analytic on $[0, s_\infty[$.

It is known (see [20]) that since $\bar{\mu}$ satisfies condition i - p and $\kappa(s) < \infty$, there exists a unique probability measure ν_s on \mathbb{P}^{d-1} such that the Radon measure $\nu_s \otimes \ell^s$ on $\mathbb{P}^{d-1} \times \mathbb{R}_+^* = (V \setminus \{0\}) / \{\pm Id\}$ satisfies

$$\bar{P}(\nu_s \otimes \ell^s) = \kappa(s) \nu_s \otimes \ell^s,$$

where, by abuse of notation, \bar{P} is the Markov operator defined by $\bar{\mu}$ on $(V \setminus \{0\}) / \{\pm Id\}$. If $\bar{x} \in \mathbb{P}^{d-1}$ corresponds to $x \in V$, we denote $|g\bar{x}| = \frac{|gx|}{|x|}$ and we consider the operator $\rho_s(\bar{\mu})$ on \mathbb{P}^{d-1} defined by

$$\rho_s(\bar{\mu})(\varphi)(\bar{x}) = \int \varphi(g \cdot \bar{x}) |g\bar{x}|^s d\bar{\mu}(g),$$

where $\bar{x} \mapsto g \cdot \bar{x}$ is the projective map defined by $g \in G$. Then ν_s is the unique probability measure on \mathbb{P}^{d-1} such that $\rho_s(\bar{\mu})\nu_s = \kappa(s)\nu_s$. Furthermore, $\text{supp}\nu_s$ is equal to the limit set of $[\text{supp}\bar{\mu}]$ and ν_s gives zero measure to any projective subspace (see [20]). The following consequence of the general renewal theorem of [32] and of the spectral gap property of the operator $\rho_s(\bar{\mu})$ plays an essential role here (see [22]).

Theorem 2.5. *If $d > 1$ and condition C holds, we have the following weak convergence:*

$$\lim_{t \rightarrow 0_+} t^{-\alpha} (t.\eta) = c(\sigma_\alpha \otimes \ell^\alpha) = \Lambda,$$

where $c > 0$, σ_α is a probability measure on \mathbb{S}^{d-1} which has projection ν_α on \mathbb{P}^{d-1} and Λ satisfies $t.\Lambda = t^\alpha \Lambda$ if $t > 0$, $\bar{P}\Lambda = \Lambda$. The above convergence is valid for any function f with a Λ -negligible set of discontinuities and such that for some $\varepsilon > 0$

$$\sup_{x \neq 0} (|x|^{-\alpha} |\log|x||^{1+\varepsilon} |f(x)|) < \infty. \quad (2.1)$$

In particular there exists $A > 0$ such that for k large enough,

$$\frac{1}{A}2^{-k\alpha} \leq \eta\{x \in V; |x| \geq 2^k\} \leq A2^{-k\alpha}.$$

Also $\Lambda(W) = 0$ for any proper affine subspace $W \subset V$.

In the special case of the recurrence relation

$$W_n = M_n^*(W_{n-1} + v) \quad (n \geq 1),$$

the corresponding measure on H is denoted by μ_v^* . The corresponding transition operator on V is denoted by T_v . Then we have the

Proposition 2.6. *Condition C is satisfied by the measure μ_v^* on H , if $v \neq 0$.*

The sequence

$$Z_n^* = \sum_{k=1}^n M_1^* \cdots M_k^*$$

converges \mathbb{P} -a.e. to

$$Z^* = \sum_{k=1}^{\infty} M_1^* \cdots M_k^*$$

where Z is defined by the \mathbb{P} -a.e convergent series $\sum_{k=1}^{\infty} M_k \cdots M_1$.

*The law η_v of Z^*v is the unique μ_v^* -stationary measure and η_v satisfies*

$$\int |x|^\theta d\eta_v(x) < \infty \quad \text{for } \theta \in [0, \alpha[, \quad \int |x|^\alpha d\eta_v(x) = \infty.$$

For any $t \in \mathbb{R}^$, we have $\eta_{tv} = t.\eta_v$. If $\alpha > 1$, for all $x \in V$ the map $v \rightarrow \eta_v(x^*)$ is a linear form.*

The Radon measure

$$\Delta_v = \lim_{t \rightarrow 0_+} t^{-\alpha}(t.\eta_v)$$

is α -homogeneous, satisfies $\Delta_{tv} = t^\alpha \Delta_v$ for $t > 0$, $\overline{P}^ \Delta_v = \Delta_v$, and Δ_{-v} is symmetric of Δ_v .*

The function $C_\alpha(v)$ satisfies for $v \neq 0$, $\text{Re} C_\alpha(v) < 0$ and for $t > 0$,

$$C_\alpha(tv) = t^\alpha C_\alpha(v) \quad \text{if } \alpha \neq 1, \quad \text{and} \quad C_1(tv) = tC_1(v) + i \langle v, \beta(t) \rangle,$$

where $\beta(t) = \int \left(\frac{tx}{1+|tx|^2} - \frac{tx}{1+|x|^2} \right) d\Lambda(x)$.

Proof. We observe that $|M^*| = |M|$, hence

$$\lim_{n \rightarrow \infty} (\mathbb{E}(|M_n^* \cdots M_1^*|^s))^{1/n} = \lim_{n \rightarrow \infty} (\mathbb{E}(|M_1 \cdots M_n|^s))^{1/n} = \kappa(s).$$

One verifies easily that condition *i-p* for $[\text{supp} \bar{\mu}]$, which is valid, remains valid for $[\text{supp} \bar{\mu}]^* = [\text{supp} \bar{\mu}^*]$. If $\text{supp} \bar{\mu}^*$ had a fixed point $x \in V$, then $g^*(x + v) = x$ for any $g \in \text{supp} \bar{\mu}$. Since v is non zero, we have $x \neq 0$. Also this implies $g_1^*(g_2^*)^{-1}x = x$ for any $g_1, g_2 \in \text{supp} \bar{\mu}$, hence x is invariant under the subgroup generated by $\text{supp} \bar{\mu}$. This contradicts irreducibility of $[\text{supp} \bar{\mu}]$. As in the proof of proposition 2.1, one sees that the condition

$$\lim_{n \rightarrow \infty} (\mathbb{E}(|M_1^* \cdots M_n^*|^\theta))^{1/n} = \kappa(\theta) < 1$$

for $\theta < \alpha$ implies the convergence

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n M_1^* \cdots M_k^* = \sum_{k=1}^{\infty} M_1^* \cdots M_k^* = Z^*.$$

Since the map $g \rightarrow g^*$ is continuous, this gives the convergence of $Z_n = \sum_{k=1}^n M_k \cdots M_1$ to $Z = \sum_{k=1}^\infty M_k \cdots M_1$.

The second assertion on η_v follows from Theorem 2.5.

The third assertion on homogeneity of η_v with respect to v follows from the relations

$$Z^*(tv) = tZ^*(v), Z^*(v+w) = Z^*(v) + Z^*(w).$$

The last assertions follow from Theorem 2.5, the relation $\eta_{tv} = t\eta_v$ for $t \in \mathbb{R}^*$ and the definition of $C_\alpha(v)$. \square

We show below the the following formula for $C_\alpha(v)$ as defined by (1.5). The proof of Theorem 1.3 will be based on the formula:

$$C_\alpha(v) = \begin{cases} \int (\mathcal{X}_v(x) - 1) \widehat{\eta}_v(x) d\Lambda(x), & \text{if } 0 < \alpha < 1; \\ \int \left((\mathcal{X}_v(x) - 1) \widehat{\eta}_v(x) - i \frac{\langle v, x \rangle}{1 + |x|^2} \right) d\Lambda(x), & \text{if } \alpha = 1; \\ \int ((\mathcal{X}_v(x) - 1) \widehat{\eta}_v(x) - i \langle v, x \rangle) d\Lambda(x), & \text{if } 1 < \alpha < 2; \\ -\frac{1}{4} \int (\langle v, w \rangle^2 + 2 \langle v, w \rangle \eta_v(w^*)) d\sigma_2(w), & \text{if } \alpha = 2. \end{cases} \quad (2.2)$$

Proposition 2.7. *The above formula for $C_\alpha(v)$ is valid.*

Proof. We suppose that C_α is defined by (2.2) and we prove that it can be transformed into (1.5). We start as in the proof of Proposition 5.19 in [5]. By definition of $\widetilde{\Lambda}$, we have

$$C_\alpha(v) = \begin{cases} \int (\widetilde{\Lambda}(y+v) - \widetilde{\Lambda}(y)) d\eta_v(y), & \text{if } \alpha \in (0, 1) \cup (1, 2]; \\ \int (\widetilde{\Lambda}(y+v) - \widetilde{\Lambda}(y)) d\eta_v(y) + i\gamma(v), & \text{if } \alpha = 1, \end{cases}$$

where $\gamma(v)$ is given by (1.6). We follow the argument in [5], but we use in an essential way the information of [20, 22], and in particular Theorem 2.5.

We define for $s < \alpha$ the Radon measure Λ_s by

$$\Lambda_s = c\sigma_s \otimes \ell^s,$$

where c is given by Theorem 2.5 and σ_s is a probability measure on \mathbb{S}^{d-1} such that

$$\bar{P}\Lambda_s = \kappa(s)\Lambda_s, \quad \text{and} \quad \lim_{s \rightarrow \alpha-} \sigma_s = \sigma_\alpha,$$

and σ_α given by Theorem 2.5. The existence of σ_s for $s < \alpha$ follows from [22]. Hence we have the weak convergence:

$$\lim_{s \rightarrow \alpha-} \Lambda_s = \Lambda_\alpha = \Lambda.$$

We define also $\widetilde{\Lambda}_s$ for $s < \alpha$, $s \neq 1$,

$$\begin{aligned} \widetilde{\Lambda}_s(y) &= \int (\mathcal{X}_y(x) - 1) d\Lambda_s(x), \quad \text{if } 0 < s < 1, \\ \widetilde{\Lambda}_s(y) &= \int (\mathcal{X}_y(x) - 1 - i \langle x, y \rangle) d\Lambda_s(x), \quad \text{if } 1 < s < 2, \end{aligned}$$

Then $\widetilde{\Lambda}_s$ depends continuously on s and $\widetilde{\Lambda}_s$ satisfies:

$$\bar{P}^* \widetilde{\Lambda}_s(x) = \int \widetilde{\Lambda}_s(g^*x) d\bar{\mu}(g) = \kappa(s) \widetilde{\Lambda}_s(x), \quad \text{and} \quad \widetilde{\Lambda}_s(tx) = t^s \widetilde{\Lambda}_s(x), \quad \text{for } t > 0.$$

For $s < \alpha$, we define

$$C_s(v) = \int (\tilde{\Lambda}_s(y+v) - \tilde{\Lambda}_s(y)) d\eta_v(y)$$

and we observe that by dominated convergence theorem,

$$\lim_{s \rightarrow \alpha_-} C_s(v) = \int (\tilde{\Lambda}(y+v) - \tilde{\Lambda}(y)) d\eta_v(y).$$

Hence $\lim_{s \rightarrow \alpha_-} C_s(v) = C_\alpha(v)$ if $\alpha \neq 1$, while $\lim_{s \rightarrow \alpha_-} C_s(v) = C_\alpha(v) - i\gamma(v)$ if $\alpha = 1$. On the other hand, $Z_0^*v = \sum_{k=0}^{\infty} M_0^* \cdots M_k^*v$ satisfies $Z_0^*v = M_0^*(Z^*v + v)$, where

$$Z^* = \sum_{k=1}^{\infty} M_1^* \cdots M_k^*$$

and M_0^* is a copy of M^* independent of Z . It follows:

$$\begin{aligned} \mathbb{E}(\tilde{\Lambda}_s(Z_0^*v)) &= \mathbb{E} \left[\int \tilde{\Lambda}_s(g^*(Z^*v + v)) d\bar{\mu}(g) \right] \\ &= \kappa(s) \mathbb{E}(\tilde{\Lambda}_s(Z^*v + v)), \end{aligned}$$

hence

$$C_s(v) = \mathbb{E}(\tilde{\Lambda}_s(Z^*v + v)) - \mathbb{E}(\tilde{\Lambda}_s(Z^*v)) = \left(\frac{1}{\kappa(s)} - 1 \right) \mathbb{E}(\tilde{\Lambda}_s(Z^*v)).$$

By Proposition 2.3, the function $\log \kappa(s)$ is convex, hence $\kappa(s)$ has a left derivative $\kappa'(\alpha_-)$ at $s = \alpha$:

$$m_\alpha = \lim_{s \rightarrow \alpha_-} \frac{1 - \kappa(s)}{\alpha - s}.$$

In order to get the value of $C_\alpha(v)$, we need to evaluate $\lim_{s \rightarrow \alpha_-} (\alpha - s) \mathbb{E}(\tilde{\Lambda}_s(Z^*v))$.

For this purpose we will use Theorem 2.5, we write

$$F_{s,v}(t) = \int_{|x| \geq t} \tilde{\Lambda}_s(\bar{x}) d\eta_v(x)$$

and we observe that $|F_{s,v}(t)| \leq \sup_{\bar{x} \in \mathbb{S}^{d-1}} |\tilde{\Lambda}_s(\bar{x})|$ is bounded by definition of $\tilde{\Lambda}_s$. Also for $t \geq 0$:

$$t^\alpha F_{s,v}(t) = \int_{|x| \geq 1} \tilde{\Lambda}_s(\bar{x}) d\eta_v^t(x)$$

with $\eta_v^t = t^{-\alpha}(t\eta_v)$. Hence, using the convergence of η_v^t to Δ_v for $t \rightarrow 0_+$ given by Theorem 2.5 and the fact that $\tilde{\Lambda}^1$ is bounded with Δ_v -negligible discontinuities, we get

$$t^\alpha F_{s,v}(t) = \Delta_v(\tilde{\Lambda}_s^1) + o(t),$$

where $o(t)$ is uniform in $s \in [0, \alpha)$ and $\tilde{\Lambda}_s^1(x) = \tilde{\Lambda}_s(\bar{x}) \mathbf{1}_{[1, \infty)}(|x|)$. By definition of $F_{s,v}$:

$$\begin{aligned} \mathbb{E}(\tilde{\Lambda}_s(Z^*v)) &= \int |y|^s \tilde{\Lambda}_s(\bar{y}) d\eta_v(y) = \int_V \left(\int_{0 < t \leq |y|} s t^{s-1} dt \right) \tilde{\Lambda}_s(\bar{y}) d\eta_v(y) \\ &= \int_0^\infty s F_{s,v}(t) t^{s-1} dt. \end{aligned}$$

Let ρ be a positive increasing function on $[0, \alpha[$ such that

$$\lim_{s \rightarrow \alpha_-} \rho(s) = +\infty, \quad \lim_{s \rightarrow \alpha_-} (\alpha - s) \rho^s(s) = 0, \quad \lim_{s \rightarrow \alpha_-} \rho^{s-\alpha}(s) = 1.$$

One can take for example $\rho(s) = (\alpha - s)^{-\frac{1}{2\alpha}}$. Then to compute the required limit, we decompose the integral of $F_{s,v}(t)$ according to the function $\rho(s)$ and use the asymptotic expansions of $F_{s,v}(t)$:

$$\begin{aligned} (\alpha - s)\mathbb{E}(\tilde{\Lambda}_s(Z^*v)) &= (\alpha - s) \int_0^{\rho(s)} sF_{s,v}(t)t^{s-1}dt \\ &+ (\alpha - s) \int_{\rho(s)}^\infty s\Delta_v(\tilde{\Lambda}_s^1)t^{-\alpha+s-1}dt + (\alpha - s) \int_{\rho(s)}^\infty o(t)t^{-\alpha+s-1}dt. \end{aligned}$$

Notice that the limits of the first and third terms are zero. Indeed, by the properties of $\rho(s)$:

$$\lim_{s \rightarrow \alpha_-} \left| (\alpha - s) \int_0^{\rho(s)} sF_{s,v}(t)t^{s-1}dt \right| \leq \lim_{s \rightarrow \alpha_-} (\alpha - s)\rho^s(s) \sup_{t>0} |F_{s,v}(t)| = 0.$$

To compute the limit of the third term, let $\epsilon > 0$ and observe that there exist $s_0 = s_0(\epsilon)$ close to α such that $o(t) < \epsilon$ for $t > \rho(s_0)$, hence using again the properties of $\rho(s)$:

$$\lim_{s \rightarrow \alpha_-} \left| (\alpha - s) \int_{\rho(s)}^\infty o(t)t^{-\alpha+s-1}dt \right| \leq \epsilon \lim_{s \rightarrow \alpha_-} \rho^{s-\alpha}(s) = \epsilon.$$

Since ϵ was arbitrary, we obtain that the limit above is in fact zero. As a result, using again the properties of $\rho(s)$,

$$\begin{aligned} \lim_{s \rightarrow \alpha_-} C_s(v) &= \lim_{s \rightarrow \alpha_-} \left(\frac{1}{\kappa(s)} - 1 \right) \mathbb{E}(\tilde{\Lambda}_s(Z^*v)) \\ &= m_\alpha \lim_{s \rightarrow \alpha_-} (\alpha - s) \int_{\rho(s)}^\infty s\Delta_v(\tilde{\Lambda}_s^1)t^{-\alpha+s-1}dt \\ &= m_\alpha \lim_{s \rightarrow \alpha_-} s\Delta_v(\tilde{\Lambda}_s^1) \lim_{s \rightarrow \alpha_-} \rho^{s-\alpha}(s) = \alpha m_\alpha \Delta_v(\tilde{\Lambda}^1), \end{aligned}$$

since $\lim_{s \rightarrow \alpha_-} \tilde{\Lambda}_s^1 = \tilde{\Lambda}^1$ and $\tilde{\Lambda}_s^1$ is uniformly bounded by a Δ_v -integrable function. The formula (1.5) for $C_\alpha(v)$ follows. \square

3 Spectral gap properties of Fourier operators, eigenfunctions and eigenvalues.

We follow closely the method of [21, 5] and we recall the corresponding functional space notations.

On continuous functions on V we introduce the semi-norm

$$[f]_{\varepsilon, \lambda} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\varepsilon (1 + |x|)^\lambda (1 + |y|)^\lambda}$$

and the two norms

$$|f|_\theta = \sup_x \frac{|f(x)|}{(1 + |x|)^\theta}, \quad ||f||_{\theta, \varepsilon, \lambda} = |f|_\theta + [f]_{\varepsilon, \lambda}.$$

Notice that the conditions $\lambda + \varepsilon \leq \theta$ (always assumed) and $[f]_{\varepsilon, \lambda} < \infty$ imply $|f|_\theta < \infty$. Define Banach spaces

$$\mathbb{C}_\theta = \{f : |f|_\theta < \infty\}, \quad \mathbb{B}_{\theta, \varepsilon, \lambda} = \{f : ||f||_{\theta, \varepsilon, \lambda} < \infty\}$$

and on them we consider the action of the transition operator P :

$$Pf(x) = \mathbb{E}(f(Mx + Q)) = \int f(hx)d\mu(h)$$

where (Q, M) is a random variable distributed according to μ . We consider also the Fourier operator P_v defined by

$$P_v f(x) = P(\mathcal{X}_v f)(x) = \mathbb{E}[\mathcal{X}_v(Mx + Q)f(Mx + Q)]$$

where $v \in V$. Notice that $P_0 = P$. We will prove later (Theorem 3.4) that the operators P_v are bounded on $\mathbb{B}_{\theta, \varepsilon, \lambda}$ for appropriately chosen parameters $\theta, \varepsilon, \lambda$. Also, for v small, they have a unique dominant eigenvalue $k(v)$ with $|k(v)| < 1$ if $v \neq 0$, $k(0) = 1$ and the rest of the spectrum of P_v is contained in a disk of center 0 and radius less than $|k(v)|$. For an operator A we denote by $\sigma(A)$ its spectrum and by $r(A)$ its spectral radius. These properties are based on the estimations below and [27, 30]. The following simple but basic fact was observed in [19]. For reader's convenience, we give its proof.

Proposition 3.1. *We have*

$$P_v^n f(x) = \mathbb{E}(\mathcal{X}_v(S_n^x) f(X_n^x)).$$

Proof. If $n = 1$, then the formula above coincide with definition of P_v . By induction, we have

$$\begin{aligned} P_v^n f(x) &= P(\mathcal{X}_v P_v^{n-1} f)(x) = \mathbb{E}[\mathcal{X}_v(Mx + Q)(P_v^{n-1} f)(Mx + Q)] \\ &= \mathbb{E}[\mathcal{X}_v(Mx + Q) \mathcal{X}_v(S_{n-1}^{Mx+Q}) f(X_{n-1}^{Mx+Q})] \\ &= \mathbb{E}[\mathcal{X}_v(S_n^x) f(X_n^x)]. \end{aligned}$$

□

The following gives the basic estimations which allow the use of [27]. Similar estimations were used in [34, 35] for different purposes.

Proposition 3.2. *There exists $D = D(\theta) < \infty$ such that for any $v \in V$, $n \in \mathbb{N}$, $\theta < \alpha$ we have*

$$|P_v^n f|_\theta \leq D |f|_\theta. \quad (3.1)$$

If $2\lambda + \varepsilon < \alpha$, $\varepsilon < 1$, $\theta < 2\lambda$, there exists constants $C_1, C_2 \geq 0, \rho \in [0, 1)$ depending on $\theta, \varepsilon, \lambda$ such that for any $n \in \mathbb{N}$, $f \in \mathbb{B}_{\theta, \varepsilon, \lambda}$, $v \in V$,

$$[P_v^n f]_{\varepsilon, \lambda} \leq C_1 \rho^n [f]_{\varepsilon, \lambda} + C_2 |v|^\varepsilon |f|_\theta. \quad (3.2)$$

Proof. Notice that

$$X_n^x = X_n^y + \Pi_n(x - y), \quad (3.3)$$

where $\Pi_n = M_n M_{n-1} \cdots M_1$. Writing $X_n = X_n^0$, by Proposition 3.1 we have

$$\begin{aligned} |P_v^n f(x)|_\theta &\leq \mathbb{E} \left[\frac{|f(X_n^x)|}{(1 + |X_n^x|)^\theta} \cdot \frac{(1 + |X_n^x|)^\theta}{(1 + |x|)^\theta} \right] \\ &\leq |f|_\theta \mathbb{E} \left[\frac{(1 + |X_n| + |\Pi_n x|)^\theta}{(1 + |x|)^\theta} \right] \\ &\leq 3^\theta |f|_\theta \mathbb{E}(1 + |X_n|^\theta + |\Pi_n|^\theta) \\ &\leq 3^\theta |f|_\theta (1 + \mathbb{E}|X_n|^\theta + C(\kappa(\theta) + \epsilon')^n) \end{aligned}$$

where $0 < \epsilon' < 1 - \kappa(\theta)$ and C is a constant. If we set $D = 3^\theta (1 + \sup_n \mathbb{E}|X_n|^\theta + C) < \infty$, the first inequality (3.1) follows.

Now we turn to the proof of (3.2). By Proposition 3.1, we have

$$P_v^n f(x) - P_v^n f(y) = \mathbb{E}[\mathcal{X}_v(S_n^x)(f(X_n^x) - f(X_n^y))] + \mathbb{E}[(\mathcal{X}_v(S_n^x) - \mathcal{X}_v(S_n^y))f(X_n^y)].$$

Without loss of generality, assume that $|x| \geq |y|$. Let

$$\begin{aligned} J_1(x, y) &= \frac{\left| \mathbb{E}[\mathcal{X}_v(S_n^x)(f(X_n^x) - f(X_n^y))] \right|}{|x - y|^\varepsilon (1 + |x|)^\lambda (1 + |y|)^\lambda}, \\ J_2(x, y) &= \frac{\left| \mathbb{E}[(\mathcal{X}_v(S_n^x) - \mathcal{X}_v(S_n^y))f(X_n^y)] \right|}{|x - y|^\varepsilon (1 + |x|)^\lambda (1 + |y|)^\lambda}. \end{aligned}$$

The first step is to estimate $J_1(x, y)$.

$$\begin{aligned}
J_1(x, y) &\leq \mathbb{E} \left(|f(X_n^x) - f(X_n^y)| / (|x - y|^\varepsilon (1 + |x|)^\lambda (1 + |y|)^\lambda) \right) \\
&\leq [f]_{\varepsilon, \lambda} \mathbb{E} \left(\frac{|X_n^x - X_n^y|^\varepsilon (1 + |X_n^x|)^\lambda (1 + |X_n^y|)^\lambda}{|x - y|^\varepsilon (1 + |x|)^\lambda (1 + |y|)^\lambda} \right) \\
&\leq [f]_{\varepsilon, \lambda} \mathbb{E} \left(\frac{|\Pi_n|^\varepsilon (1 + |X_n| + |\Pi_n x|)^\lambda (1 + |X_n| + |\Pi_n y|)^\lambda}{(1 + |x|)^\lambda (1 + |y|)^\lambda} \right) \\
&\leq [f]_{\varepsilon, \lambda} \mathbb{E} (|\Pi_n|^\varepsilon (1 + |X_n| + |\Pi_n|)^{2\lambda}) \\
&\leq 3^{2\lambda} [f]_{\varepsilon, \lambda} \left(\mathbb{E} |\Pi_n|^\varepsilon + \mathbb{E} |\Pi_n|^{2\lambda + \varepsilon} + (\mathbb{E} |\Pi_n|^{2\lambda + \varepsilon})^{\frac{\varepsilon}{2\lambda + \varepsilon}} (\mathbb{E} |X_n|^{2\lambda + \varepsilon})^{\frac{2\lambda}{2\lambda + \varepsilon}} \right).
\end{aligned}$$

Proposition 2.3 allows us to choose $\epsilon_1 > 0$ and a constant A_1 such that

$$\max\{\kappa(\varepsilon), \kappa(2\lambda + \varepsilon)\} + \epsilon_1 < 1,$$

and for all $n \in \mathbb{N}$,

$$\mathbb{E} |\Pi_n|^{2\lambda + \varepsilon} \leq A_1 (\kappa(2\lambda + \varepsilon) + \epsilon_1)^n, \quad \mathbb{E} |\Pi_n|^\varepsilon \leq A_1 (\kappa(\varepsilon) + \epsilon_1)^n.$$

Now setting

$$\rho = \max \left\{ \kappa(\varepsilon) + \epsilon_1, \kappa(2\lambda + \varepsilon) + \epsilon_1, (\kappa(2\lambda + \varepsilon) + \epsilon_1)^{\frac{2\lambda}{2\lambda + \varepsilon}} \right\}$$

and

$$C_1 = 3^{2\lambda} \left(2A_1 + (A_1)^{\frac{\varepsilon}{2\lambda + \varepsilon}} \sup_n (\mathbb{E} |X_n|^{2\lambda + \varepsilon})^{\frac{2\lambda}{2\lambda + \varepsilon}} \right),$$

we have

$$J_1(x, y) \leq C_1 \rho^n [f]_{\varepsilon, \lambda}. \quad (3.4)$$

Now we are going to estimate $J_2(x, y)$. Observe that

$$|e^{i\langle x, y \rangle} - 1| \leq 2|x|^\varepsilon |y|^\varepsilon \quad \text{and} \quad S_n^x - S_n^y = Z_n(x - y),$$

where $Z_n = \sum_{k=1}^n M_k \cdots M_1$. Using these facts, we get

$$\begin{aligned}
J_2(x, y) &\leq 2|v|^\varepsilon |f|_\theta \mathbb{E} \left[\frac{|Z_n|^\varepsilon (1 + |X_n^y|)^\theta}{(1 + |x|)^\lambda (1 + |y|)^\lambda} \right] \\
&\leq 2|v|^\varepsilon |f|_\theta \mathbb{E} \left[\frac{|Z_n|^\varepsilon (1 + |X_n| + |\Pi_n| \cdot |y|)^\theta}{(1 + |x|)^\lambda (1 + |y|)^\lambda} \right] \\
&\leq 2 \cdot 3^\theta |v|^\varepsilon |f|_\theta \mathbb{E} [|Z_n|^\varepsilon (1 + |X_n|^\theta + |\Pi_n|^\theta)]
\end{aligned}$$

To finish our proof, the left thing is to prove the finiteness of the expectation in the last expression.

For $s < \alpha$, by the properties of $\kappa(s)$, there exists $\epsilon_s > 0$ and a constant $A_s > 1$ such that

$$\kappa(s) + \epsilon_s < 1 \quad \text{and} \quad \mathbb{E} |\Pi_n|^s \leq A_s (\kappa(s) + \epsilon_s)^n.$$

Then if $s < \min\{1, \alpha\}$,

$$\mathbb{E} |Z_n|^s \leq 1 + \sum_{m=1}^n \mathbb{E} |\Pi_m|^s \leq A_s \sum_{m=0}^n (\kappa(s) + \epsilon_s)^m,$$

and if $s \in [1, \alpha)$,

$$\mathbb{E} |Z_n|^s \leq \left(1 + \sum_{m=1}^n (\mathbb{E} |\Pi_m|^s)^{\frac{1}{s}} \right)^s \leq \left(A_s \sum_{m=0}^n (\kappa(s) + \epsilon_s)^{\frac{m}{s}} \right)^s.$$

Therefore for $s < \alpha$,

$$\sup_n \mathbb{E}|Z_n|^s < \infty.$$

Also we have that $\sup_n \mathbb{E}|X_n|^q < \infty$ for $q < \alpha$. Now noticing that $\theta + \varepsilon < \alpha$ and applying the Hölder inequality, we obtain that

$$\sup_n \mathbb{E}[|Z_n|^\varepsilon (1 + |X_n|^\theta + |\Pi_n|^\theta)] < \infty.$$

We set $C_2 = 2 \cdot 3^\theta \sup_n \mathbb{E}[|Z_n|^\varepsilon (1 + |X_n|^\theta + |\Pi_n|^\theta)]$ and thus

$$J_2(x, y) \leq C_2 |v|^\varepsilon |f|_\theta. \quad (3.5)$$

Finally combining (3.4) and (3.5) we obtain that

$$[P_v^n f]_{\varepsilon, \lambda} \leq \sup_{x, y} (J_1(x, y) + J_2(x, y)) \leq C_1 \rho^n [f]_{\varepsilon, \lambda} + C_2 |v|^\varepsilon |f|_\theta.$$

□

Proposition 3.3. *For any $v \neq 0$, the equation $P_v f = zf$, $|z| = 1$, $f \in \mathbb{B}_{\theta, \varepsilon, \lambda}$ implies $f = 0$. In particular, $r(P_v) < 1$.*

If $\text{supp} \bar{\mu}$ consists of similarities, this is Lemma 3.14 in [5]; in view of its role here we give the proof.

Proof of Proposition 3.3. Assume that $P_v f = zf$ for some nonzero $f \in \mathbb{B}_{\theta, \varepsilon, \lambda}$. Then the function f is bounded. Indeed for every n

$$|f(x)| = |z^n f(x)| \leq P^n(|f|)(x),$$

hence

$$|f(x)| \leq \lim_{n \rightarrow \infty} P^n(|f|)(x) = \eta(|f|).$$

Next observe that since f is continuous, on the support of η the function $|f|$ is equal to its maximum and without loss of generality we may assume that this maximum is 1. For every n and $x \in \text{supp} \eta$, noticing that $z^n f(x) = \mathbb{E}[e^{i\langle v, S_n^x \rangle} f(X_n^x)]$ and using a convexity argument, we can show that

$$z^n f(x) = e^{i\langle v, S_n^x \rangle} f(X_n^x) \quad \mathbb{P}\text{-a.e.}$$

Hence for every $x, y \in \text{supp} \eta$,

$$\frac{f(x)}{f(y)} e^{i\langle v, Z_n(y-x) \rangle} = \frac{f(X_n^x)}{f(X_n^y)}, \quad (3.6)$$

where $Z_n = \sum_{k=1}^n M_k \cdots M_1$. By the Hölder inequality, we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbb{E} \left| \frac{f(X_n^x)}{f(X_n^y)} - 1 \right| \\ & \leq [f]_{\varepsilon, \lambda} \limsup_{n \rightarrow \infty} \mathbb{E}[|X_n^x - X_n^y|^\varepsilon (1 + |X_n^x|)^\lambda (1 + |X_n^y|)^\lambda] \\ & = [f]_{\varepsilon, \lambda} \limsup_{n \rightarrow \infty} \mathbb{E}[|M_n \cdots M_1(x - y)|^\varepsilon (1 + |X_n^x|)^\lambda (1 + |X_n^y|)^\lambda] \\ & \leq [f]_{\varepsilon, \lambda} |x - y|^\varepsilon \limsup_{n \rightarrow \infty} [\mathbb{E}|M_n \cdots M_1|^{2\lambda + \varepsilon}]^{\frac{\varepsilon}{2\lambda + \varepsilon}} \cdot \limsup_{n \rightarrow \infty} [\mathbb{E}(1 + |X_n^x|)^{\lambda + \frac{\varepsilon}{2}} (1 + |X_n^y|)^{\lambda + \frac{\varepsilon}{2}}]^{\frac{2\lambda}{2\lambda + \varepsilon}} \end{aligned}$$

By our assumption, the first limit is zero and the second one is finite. Hence

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left| \frac{f(X_n^x)}{f(X_n^y)} - 1 \right| = 0$$

Therefore for \mathbb{P} a.e. trajectory ω there exists a sequence $\{n_k\} = \{n_k(\omega)\}$ such that

$$\lim_{n_k \rightarrow \infty} \frac{f(X_{n_k}^x)}{f(X_{n_k}^y)} = 1.$$

By Proposition 2.6, $\lim_{n \rightarrow \infty} Z_n(\omega) = Z(\omega)$ exists a.s.. Hence letting $k \rightarrow \infty$ we obtain that there is Ω_0 such that $\mathbb{P}(\Omega_0) = 1$ and for $\omega \in \Omega_0$,

$$\frac{f(x)}{f(y)} = e^{i\langle v, Z(\omega)(x-y) \rangle}.$$

We are going to prove that this leads to a contradiction whenever $v \neq 0$. We choose $x_j, y_j \in \text{supp}\eta$, $j = 1, \dots, d$ with $x_j - y_j$ spanning V as a vector space. Such points exist because the support of η , as a set invariant under the action of $\text{supp}\mu$, is not contained in some proper affine subspace of V . Let η_v be the law of $W(\omega) = Z^*(\omega)v$. Then for every j the support of η_v is contained in the union of affine hyperplanes $\bigcup_{n \in \mathbb{Z}} \{H_j + ns_j v_j\}$, where H_j is some hyperplane orthogonal to v_j and s_j is appropriately chosen constants. Taking intersection of all such sets defined for every j we conclude that $\text{supp}\eta_v$ is contained in some discrete set of points, hence $\text{supp}\eta_v$ is discrete. This contradicts Proposition 2.3.

For the last assertion we observe that in view of Theorem of Ionescu Tulcea and Marinescu [27], if z belongs to the spectrum of P_v and $|z| = 1$ then z is an eigenvalue of P_v . \square

The following corresponds to Theorem 1.2 and is our basic tool for the study of P_v .

Theorem 3.4. *Assume $\theta, \varepsilon, \lambda$ satisfy $0 < \varepsilon < 1$, $2\lambda + \varepsilon < \alpha$, $\theta \leq 2\lambda$. Then P_v has the following properties:*

- 1) P_v is a bounded operator on $\mathbb{B}_{\theta, \varepsilon, \lambda}$ with spectral radius $r(P_v) \leq 1$;
- 2) If $v \neq 0$, $r(P_v) < 1$;
- 3) If $v = 0$, $P = P_0$ satisfies $P\mathbf{1} = \mathbf{1}$, $P\eta = \eta$. The operator Q on $\mathbb{B}_{\theta, \varepsilon, \lambda}$ defined by $Qf = Pf - \eta(f)\mathbf{1}$ has spectral radius less than 1 and $\eta(Qf) = 0$. In other words, P is the direct sum of the Identity on $\mathbb{C}\mathbf{1}$ and of an operator on $\text{Ker}\eta$ with spectral radius strictly less than 1.

Proof of Theorem 3.4. Proposition 3.2 implies that P_v is a power-bounded operator on $\mathbb{B}_{\theta, \varepsilon, \lambda}$, hence assertion 1 follows. Since bounded subsets of $(\mathbb{B}_{\theta, \varepsilon, \lambda}, \|\cdot\|_{\theta, \varepsilon, \lambda})$ are relatively compact in $(\mathbb{C}_\theta, |\cdot|_\theta)$, the inequality in part 2 of Proposition 3.2 shows that we can apply the theorem of Ionescu-Tulcea and Marinescu (see [27]) to P_v . In particular, if for some $v \in V$, $r(P_v) = 1$, there exists $f \in \mathbb{B}_{\theta, \varepsilon, \lambda}$ and $z \in \mathbb{C}, |z| = 1$, $f \neq 0$ such that $P_v f = zf$. If $v \neq 0$, this contradicts Proposition 3.3, hence assertion 2 follows.

If $v = 0$, part 2 of Proposition 3.2 gives: $[P^{n_0} f]_{\varepsilon, \lambda} \leq \rho_1 [f]_{\varepsilon, \lambda}$ for some $n_0 \in \mathbb{N}, \rho_1 \in [0, 1[$. We show that $f \rightarrow [f]_{\varepsilon, \lambda}$ defines a norm equivalent to $f \rightarrow |f|_\theta$ on the subspace $\text{Ker}\eta = \{f \in \mathbb{B}_{\theta, \varepsilon, \lambda}; \eta(f) = 0\}$. Since $\eta(f) = 0$, if $f \in \text{Ker}\eta$, the condition $[f]_{\varepsilon, \lambda} = 0$ implies $f = 0$. Hence $f \rightarrow [f]_{\varepsilon, \lambda}$ is a norm on $\text{Ker}\eta$, which satisfies $[f]_{\varepsilon, \lambda} \leq \|f\|_{\theta, \varepsilon, \lambda}$. Since $\varepsilon \leq 1$ we have

$$\begin{aligned} |f(x) - f(y)| &\leq [f]_{\varepsilon, \lambda} |x - y|^\varepsilon (1 + |x|^\lambda)(1 + |y|^\lambda) \\ &\leq 4[f]_{\varepsilon, \lambda} (1 + |x|^{\lambda+\varepsilon})(1 + |y|^{\lambda+\varepsilon}). \end{aligned}$$

Since $\lambda + \varepsilon < \theta < \alpha$, we have $1 + |x|^{\lambda+\varepsilon} \leq 2(1 + |x|^\theta)$ and $\int |y|^{\lambda+\varepsilon} d\eta(y) = D < \infty$. Hence, using $\eta(f) = 0$:

$$|f(x)| \leq 8(1 + D)[f]_{\varepsilon, \lambda}(1 + |x|^\theta),$$

i.e. $|f|_\theta \leq 8(1 + D)[f]_{\varepsilon, \lambda}$. The equivalence of norms follows.

We can write $\mathbb{B}_{\theta,\varepsilon,\lambda} = \mathbb{C}\mathbf{1} \oplus \text{Ker}\eta$. Since $P\mathbf{1} = \mathbf{1}$ and $P\eta = \eta$, the subspaces $\mathbb{C}\mathbf{1}$ and $\text{Ker}\eta$ are closed P -invariant subspaces of $\mathbb{B}_{\theta,\varepsilon,\lambda}$. Since $Q\mathbf{1} = 0$, Q can be identified with its restriction to $\text{Ker}\eta$. Then the inequality $[Q^{n_0}f]_{\varepsilon,\lambda} \leq \rho_1[f]_{\varepsilon,\lambda}$ and the equivalence of norms observed above imply

$$r(Q^{n_0}) \leq \rho_1, \quad r(Q) \leq \rho_1^{1/n_0} < 1.$$

□

The study of P_{tv} for t small and v fixed is based on a theorem of Keller and Liverani([30]), Proposition 3.2 and the following easy lemma.

Lemma 3.5. *If $\lambda + 2\varepsilon < \theta < \alpha$, $\delta \leq \varepsilon$, there exists $C > 0$ such that for any $\gamma \in [\lambda + 2\varepsilon, \theta]$ and $v, w \in V$:*

$$|(P_v - P_w)f|_\gamma \leq C|v - w|^\delta \|f\|_{\theta,\varepsilon,\lambda}.$$

Proof. We observe that

$$\begin{aligned} |(P_v - P_w)f(x)| &\leq \int |e^{i\langle v, hx \rangle} - e^{i\langle w, hx \rangle}| |f(hx)| d\mu(h) \\ &\leq 2|v - w|^\delta \int |hx|^\delta |f(hx) - f(0)| d\mu(h) + 2|v - w|^\delta |f(0)| \int |hx|^\delta d\mu(h) \\ &\leq 2|v - w|^\delta [f]_{\varepsilon,\lambda} \int |hx|^{\delta+\varepsilon} (1 + |hx|)^\lambda d\mu(h) + 2|v - w|^\delta |f|_\theta \int |hx|^\delta d\mu(h). \end{aligned}$$

Therefore if we take $C = \sup_x \{2 \int [|hx|^{\delta+\varepsilon} (1 + |hx|)^\lambda + |hx|^\delta] d\mu(h) / (1 + |x|)^{\lambda+2\varepsilon}\}$, then

$$|(P_v - P_w)f|_\gamma = \sup_x |(P_v - P_w)f(x) / (1 + |x|)^\gamma| \leq C|v - w|^\delta \|f\|_{\theta,\varepsilon,\lambda}.$$

□

In view of Theorem 3.4 and Lemma 3.5, we may use the perturbation theorem of [30] for the family P_{tv} , hence as in [21, 5] we have the following

Proposition 3.6. *Assume $\varepsilon < 1$, $\lambda + 2\varepsilon < \theta < 2\lambda + \varepsilon < \alpha$, $v \in V$. Then there exists $t_0 > 0$, $\delta > 0$, $\rho < 1 - \delta$ such that for every $t \in \mathbb{R}$ with $|t| \leq t_0$:*

- a) *The spectrum of P_{tv} acting on $\mathbb{B}_{\theta,\varepsilon,\lambda}$ is contained in $\mathfrak{S} = \{z \in \mathbb{C}; |z| \leq \rho\} \cup \{z \in \mathbb{C}; |z - 1| < \delta\}$.*
- b) *The set $\sigma(P_{tv}) \cap \{z \in \mathbb{C}; |z - 1| \leq \delta\}$ consists of exactly one eigenvalue $k(tv)$, the corresponding eigenspace is one dimensional and $\lim_{t \rightarrow 0} k(tv) = 1$.*
- c) *If π_{tv} is the spectral projection on the above eigenspace of P_{tv} , there exists an operator Q_{tv} with $r(Q_{tv}) \leq \rho$, $\pi_{tv}Q_{tv} = Q_{tv}\pi_{tv} = 0$ and for every $n \in \mathbb{N}$, $f \in \mathbb{B}_{\theta,\varepsilon,\lambda}$,*

$$P_{tv}^n f = k^n(tv) \pi_{tv}(f) + Q_{tv}^n(f).$$

Furthermore $k(tv)$, π_{tv} , Q_{tv} depends continuously on t .

- d) *For any z in the complement of \mathfrak{S} :*

$$\|(z - P_{tv})^{-1}f\|_{\theta,\varepsilon,\lambda} \leq D\|f\|_{\theta,\varepsilon,\lambda}$$

for some constant D independent of t .

This statement allow us to complete the proof of Theorem 1.2. For t small define the function $g_{tv} = \pi_{tv}(\mathbf{1})$. Hence

$$P_{tv}g_{tv} = k(tv)g_{tv}.$$

Then for any function f in $\mathbb{B}_{\theta,\varepsilon,\lambda}$ we define $\mathcal{E}_{tv}(f) \in \mathbb{C}$ by $\pi_{tv}(f) = \mathcal{E}_{tv}(f)g_{tv}$.

We will be able to get the asymptotic expression of $k(tv)$ for t small through the use of a new family of operators $T_{t,v}$ on $\mathbb{B}_{\theta,\varepsilon,\lambda}$ defined by

$$T_{t,v}f(x) = \int \mathcal{X}_{tb}(x+v)f(g^*(x+v))d\mu(h).$$

Then $T_v = T_{0,v}$, $T_v\eta_v = \eta_v$, where η_v is the stationary measure for the Markov chain W_n . It turns out that the analogues of Theorem 3.4, Proposition 3.6, are valid for the family $T_{t,v}$. Therefore, for small values of t , the spectrum of $T_{t,v}$ in some neighborhood of 1 consists of only one point $k^*(t,v)$ which satisfies $|k^*(t,v)| = r(T_{t,v})$. We denote by $T_{t,v}^*$ the dual operator on $\mathbb{B}_{\theta,\varepsilon,\lambda}^*$ of $T_{t,v}$. One observes that for any $v \in V$, the function \mathcal{X}_v belongs to $\mathbb{B}_{\theta,\varepsilon,\lambda}$ and $\|\mathcal{X}_v\|_{\theta,\varepsilon,\lambda} \leq 1 + 2|v|^\varepsilon$. It follows that for any $\mathcal{E} \in \mathbb{B}_{\theta,\varepsilon,\lambda}^*$,

$$\widehat{\mathcal{E}}(v) = \mathcal{E}(\mathcal{X}_v)$$

plays the role of a Fourier transform for \mathcal{E} and

$$|\widehat{\mathcal{E}}(v)| \leq (1 + 2|v|^\varepsilon)\|\widehat{\mathcal{E}}\|_{\theta,\varepsilon,\lambda}.$$

The following relation between P_{tv} and $T_{t,v}$ plays an essential role in the calculation of the asymptotic expansion for $k(tv)$.

Proposition 3.7. *For any $t \in \mathbb{R}$, $v \in V \setminus \{0\}$, $\mathcal{E} \in \mathbb{B}_{\theta,\varepsilon,\lambda}^*$,*

$$P_{tv}(\widehat{\mathcal{E}} \circ t) = (\widehat{T_{t,v}^* \mathcal{E}}) \circ t.$$

Proof. As in [21], the proof is based on the definitions of \mathcal{X}_x , $T_{t,v}$ and the fact that the map $x \rightarrow tx$ commute with $x \rightarrow gx$ for $g \in G$. However, in view of its role here, we give it explicitly. Since $x \rightarrow gx$ ($g \in G$) and $x \rightarrow tx$ commute:

$$\begin{aligned} T_{t,v}(\mathcal{X}_{tx})(y) &= \int \mathcal{X}_{tb}(y+v)\mathcal{X}_{tx}(g^*(y+v))d\mu(h) \\ &= \int \mathcal{X}_{tb}(y+v)\mathcal{X}_{t(gx)}(y+v)d\mu(h) = \int \mathcal{X}_{t(gx+b)}(y+v)d\mu(h); \\ P_{tv}(\widehat{\mathcal{E}} \circ t)(x) &= \iint \mathcal{X}_{tv}(gx+b)\mathcal{X}_y(t(gx+b))d\mathcal{E}(y)d\mu(h) \\ &= \iint \mathcal{X}_{y+v}(t(gx+b))d\mathcal{E}(y)d\mu(h) = \mathcal{E}(T_{t,v}(\mathcal{X}_{tx})) = \widehat{T_{t,v}^* \mathcal{E}}(tx). \end{aligned}$$

□

As in [21], this proposition allows us to construct an eigenfunction of P_{tv} from an eigenfunctional $\eta_{t,v}$ of $T_{t,v}$, hence in Section 4 it will lead to the expansion of $k(tv)$ at $t = 0$, using the following result (see [21, Corollary 2]):

Corollary 3.8. *Assume $\eta_{t,v} \in \mathbb{B}_{\theta,\varepsilon,\lambda}^*$ satisfies*

$$T_{t,v}^*\eta_{t,v} = k^*(t,v)\eta_{t,v}, \quad \eta_{t,v}(\mathbf{1}) = 1.$$

If $\varepsilon < 1/2$, there exists $t_3 > 0$ such that if $|t| \leq t_3$, the function

$$\psi_{tv} = \widehat{\eta_{t,v}} \circ t$$

is the unique normalized eigenfunction of P_{tv} (with value 1 at 0) acting on $\mathbb{B}_{\theta,\varepsilon,\lambda}$ and corresponding to the eigenvalue $k(tv)$, i.e.

$$P_{tv}(\psi_{tv}) = k(tv)\psi_{tv}, \quad \psi_{tv}(0) = 1.$$

Moreover $k(tv) = k^(t,v)$ and*

$$(k(tv) - 1)\eta(\psi_{tv}) = \eta(\psi_{tv}(\mathcal{X}_{tv} - 1)).$$

Remark 3.9. In particular, using assertion c of Proposition 3.6, we see that $\lim_{t \rightarrow 0} \|\psi_{tv} - 1\|_{\theta, \varepsilon, \lambda} = 0$. Since η defines an element of $\mathbb{B}_{\theta, \varepsilon, \lambda}$, we have $\lim_{t \rightarrow 0} \eta(\psi_{t, v}) = 1$.

4 Asymptotic expansion of eigenvalues in terms of tails and the proof of Theorem 1.1

Using the techniques of [30, 35] and the above results, we deduce from Proposition 3.6 the following result (see [5, Proposition 3.18]):

Proposition 4.1. *Assume additionally that $\lambda + 3\varepsilon < \theta$, $2\lambda + 3\varepsilon < \alpha$. Then the identity embedding of $\mathbb{B}_{\theta, \varepsilon, \lambda}$ into $\mathbb{B}_{\theta, \varepsilon, \lambda + \varepsilon}$ is continuous and the decomposition $P_{tv} = k(tv)\pi_{tv} + Q_{tv}$ coincide on both spaces. Moreover, there exists constants $D > 0$ and $t_1 > 0$ such that for $|t| \leq t_1$, we have if $|v| \leq 1$:*

- (i) $\|(P_{tv} - P)f\|_{\theta, \varepsilon, \lambda + \varepsilon} \leq D|t|^\varepsilon \|f\|_{\theta, \varepsilon, \lambda}$;
- (ii) $\|(k(tv)\pi_{tv} - \pi_0)f\|_{\theta, \varepsilon, \lambda + \varepsilon} \leq D|t|^\varepsilon \|f\|_{\theta, \varepsilon, \lambda}$;
- (iii) $\|(\pi_{tv} - \pi_0)f\|_{\theta, \varepsilon, \lambda + \varepsilon} \leq D|t|^\varepsilon \|f\|_{\theta, \varepsilon, \lambda}$;
- (iv) $\|(Q_{tv} - Q)f\|_{\theta, \varepsilon, \lambda} \leq D|t|^\varepsilon \|f\|_{\theta, \varepsilon, \lambda}$;
- (v) $\|g_{tv} - \mathbf{1}\|_{\theta, \varepsilon, \lambda} \leq D|t|^\varepsilon$;
- (vi) $|k(tv) - 1| \leq D|t|^\varepsilon$;
- (vii) \mathcal{E}_{tv} is a bounded functional on $\mathbb{B}_{\theta, \varepsilon, \lambda}$ with norm at most $D|t|^\varepsilon$.

The following is a consequence of Proposition 3.8, 4.1 and the homogeneity at infinity of stationary measures, given by Theorem 2.5. It is a detailed form of Theorem 1.3.

Theorem 4.2. *a) If $0 < \alpha < 1$, then*

$$\lim_{t \rightarrow 0_+} \frac{k(tv) - 1}{t^\alpha} = C_\alpha(v)$$

with

$$C_\alpha(v) = \int (\mathcal{X}_v(x) - 1) \widehat{\eta}_v(x) d\Lambda(x).$$

b) If $\alpha = 1$, then

$$\lim_{t \rightarrow 0_+} \frac{k(tv) - 1 - i \langle v, \delta(t) \rangle}{t} = C_1(v)$$

with

$$C_1(v) = \int_V \left((\mathcal{X}_v(x) - 1) \widehat{\eta}_v(x) - i \frac{\langle v, x \rangle}{1 + |x|^2} \right) d\Lambda(x)$$

and $\delta(t) = \int_V \frac{tx}{1 + |tx|^2} d\eta(x)$. Furthermore there exists a constant K_\dagger with $K_\dagger = c_1 + 4A/\log 2 + \int_{|x|>1} |x|/(1 + |x|^2) d\Lambda(x)$ (A is given in Theorem 2.5) such that

$$|\delta(t)| \leq \begin{cases} K_\dagger |t| |\log t|, & |t| \leq 1/2, \\ K_\dagger |t|, & |t| > 1/2. \end{cases}$$

c) If $1 < \alpha < 2$,

$$\lim_{t \rightarrow 0_+} \frac{k(tv) - 1 - i \langle v, tm \rangle}{t^\alpha} = C_\alpha(v)$$

where

$$C_\alpha(v) = \int ((\mathcal{X}_v(x) - 1) \widehat{\eta}_v(x) - i \langle v, x \rangle) d\Lambda(x).$$

d) If $\alpha = 2$,

$$\lim_{t \rightarrow 0} \frac{k(tv) - 1 - i \langle v, tm \rangle}{t^2 |\log |t||} = 2C_2(v)$$

where

$$C_2(v) = -\frac{1}{4} \int (\langle v, w \rangle^2 + 2 \langle v, w \rangle \eta_v(w^*)) d\sigma_2(w)$$

is a quadratic form.

e) If $\alpha > 2$, then

$$\lim_{t \rightarrow 0} \frac{k(tv) - 1 - i \langle v, tm \rangle}{t^2} = C_{2+}(v)$$

with

$$C_{2+}(v) = -\frac{1}{2} q(v, v) - q(v, (I - z^*)^{-1} z^* v).$$

The proof is based on estimations of P_{tv} , ψ_{tv} , η , which are valid here, as in [5, Theorem 5.1]; these estimations are formal consequences of the statements in Theorem 2.5, Corollary 3.8, which in turn correspond to relations (2.2), (2.3) and Lemma 3.23 of [5].

To prove our theorem 4.2, we need further properties of the stationary measure η . In particular, essential use is made of the homogeneity at infinity of η stated in Theorem 2.5. We will deal with expressions of the form $\int_V f(t, x) d\eta(x)$ for $t \in \mathbb{R}$ and we are interested in their behavior for small values of $|t|$. We denote by I_1 the interval $[-1, 1]$. Now we give some technical lemmas which will be used often later. For the proof of these lemmas, see [5, section 4].

Lemma 4.3. *Let f be any continuous function on $I_1 \times V$ satisfying*

$$|f(t, x)| \leq \begin{cases} D_{\delta, \beta} |t|^{\delta+\beta} |x|^\beta, & \text{for } |tx| > 1; \\ D_{\delta, \gamma} |t|^{\delta+\gamma} |x|^\gamma, & \text{for } |tx| \leq 1, \end{cases} \quad (4.1)$$

where $\beta < \alpha$, $\gamma + \delta > \alpha$ and $\delta > 0$. Then

$$\lim_{t \rightarrow 0} \frac{1}{|t|^\alpha} \int_V f(t, x) d\eta(x) = 0.$$

Now we present some properties of the eigenfunction ψ_{tv} . To do this, we will need some further hypotheses on the parameters $\theta, \varepsilon, \lambda$ and from now on, we will assume additionally that

$$\begin{aligned} \text{if } 1 < \alpha < 2, & \quad \text{then } 1 + \lambda + \varepsilon > \alpha, \\ \text{if } \alpha = 2, & \quad \text{then } \lambda + 2\varepsilon > 1, \\ \text{if } \alpha > 2, & \quad \text{then } \lambda = 1. \end{aligned}$$

It is easy to prove that there exists $\theta, \varepsilon, \lambda$ satisfying all the assumptions in our theorems and the conditions above.

Lemma 4.4. *There exists D'' such that*

$$\begin{aligned} |\psi_{tv}(x) - \hat{\eta}_v(tx)| &\leq D'' |t|^{2\varepsilon} |x|^\varepsilon, & \text{for } |tx| > 1; \\ |\psi_{tv}(x) - \hat{\eta}_v(tx)| &\leq D'' |t|^\varepsilon |tx|^\tau, & \text{for } |tx| \leq 1, \end{aligned}$$

for $\tau = \min\{1, \lambda + \varepsilon\}$.

Corollary 4.5. *If $\alpha \leq 2$, then*

$$\lim_{t \rightarrow 0} \frac{1}{|t|^\alpha} \int_V (\mathcal{X}_v(tx) - 1) (\psi_{tv}(x) - \hat{\eta}_v(tx)) d\eta(x) = 0.$$

We will need also the speed of convergence of $\eta(\psi_{tv})$ to 1.

Lemma 4.6. *Assume v is fixed. Then there exists $D''' > 0$ and $t_3 > 0$ such that for $|t| < t_3$, we have*

$$|1 - \eta(\psi_{tv})| \leq D''' |t|^{\min\{1, \lambda + \varepsilon\}}.$$

As an example of how to use the above estimations and the basic Theorem 2.5, let us consider in more detail the cases $\alpha < 1$ and $\alpha = 1$. For the cases $\alpha \in]1, 2]$, $\alpha > 2$ we refer to [5, section 5].

Proof of Theorem 4.2. Case $\alpha < 1$. We use the expression of $\psi_{tv}, k(tv)$ given by Corollary 3.8 and write for $t > 0$,

$$\begin{aligned} \frac{1}{t^\alpha}(k(tv) - 1)\eta(\psi_{tv}) &= \frac{1}{t^\alpha} \int (\mathcal{X}_v(tx) - 1)\psi_{tv}(x)d\eta(x) \\ &= \frac{1}{t^\alpha} \int (\mathcal{X}_v(tx) - 1)\widehat{\eta}_v(tx)d\eta(x) + \frac{1}{t^\alpha} \int (\mathcal{X}_v(tx) - 1)(\psi_{tv}(x) - \widehat{\eta}_v(tx))d\eta(x). \end{aligned}$$

We observe that the function $f_v = (\mathcal{X}_v - 1)\widehat{\eta}_v$ satisfies the regularity and growth conditions of Theorem 2.5 since $f_v(x)$ is bounded and $|f_v(x)| \leq 2|x|$ for $|x| \leq 1$. Hence the first term converges to

$$\int (\mathcal{X}_v(x) - 1)\widehat{\eta}_v(x)d\Lambda(x).$$

The use of Corollary 4.5 shows that the second term has limit zero, hence the result follows from Remark 3.9.

Case $\alpha = 1$. Using Corollary 3.8, we see that

$$\begin{aligned} t^{-1}[k(tv) - 1 - i\langle v, \delta(t) \rangle] &= [t\eta(\psi_{tv})]^{-1} \left[\left(\eta(\psi_{tv}(\mathcal{X}_{tv} - 1)) - i\langle v, \delta(t) \rangle \right) + i(1 - \eta(\psi_{tv}))\langle v, \delta(t) \rangle \right] \\ &= [\eta(\psi_{tv})]^{-1} [K_{11}(t) + K_{12}(t) + K_{13}(t)], \end{aligned}$$

where

$$\begin{aligned} K_{11}(t) &= t^{-1} \int_V \left(\widehat{\eta}_v(tx)(\mathcal{X}_{tv}(x) - 1) - it \frac{\langle v, x \rangle}{1 + |tx|^2} \right) d\eta(x), \\ K_{12}(t) &= t^{-1} \int_V (\psi_{tv}(x) - \widehat{\eta}_v(tx))(\mathcal{X}_{tv}(x) - 1) d\eta(x), \\ K_{13}(t) &= it^{-1} (1 - \eta(\psi_{tv})) \langle v, \delta(t) \rangle. \end{aligned}$$

By Corollary 4.5,

$$\lim_{t \rightarrow 0_+} K_{12}(t) = 0. \quad (4.2)$$

Next observe that the function $f_1(x) = \widehat{\eta}_v(x)(\mathcal{X}_v(x) - 1) - i \frac{\langle v, x \rangle}{1 + |x|^2}$ satisfies the growth condition (2.1) in Theorem 2.5. Indeed f_1 is bounded and for $|x| \leq 1$,

$$\begin{aligned} |f_1(x)| &= |(\widehat{\eta}_v(x) - 1)(\mathcal{X}_v(x) - 1)| + |\mathcal{X}_v(x) - 1 - i \frac{\langle v, x \rangle}{1 + |x|^2}| \\ &\leq 2|v| \cdot |x| \cdot \|\mathcal{X}_x - 1\|_{\theta, \varepsilon, \lambda} + 4(|v| \cdot |x|)^2 \\ &\leq 8|v| \cdot |x|^{1+\lambda+\varepsilon} + 4(|v| \cdot |x|)^2, \end{aligned}$$

where in the last step, we use the estimation

$$\|\mathcal{X}_x - 1\|_{\theta, \varepsilon, \lambda} \leq 4|x|^{\min\{1, \lambda+\varepsilon\}},$$

which can be shown by direct calculation. Thus by Theorem 2.5, we have that

$$\lim_{t \rightarrow 0_+} K_{11}(t) = \Lambda(f_1) = C_1(v). \quad (4.3)$$

Now the left thing is to evaluate the term $K_{13}(t)$.

We first need to show the following properties of $\delta(t)$:

$$|\delta(t)| \leq \begin{cases} K_{\dagger}|t|, & |t| \geq \frac{1}{2}; \\ K_{\dagger}|t \log |t||, & |t| < \frac{1}{2}, \end{cases} \quad (4.4)$$

with $K_{\dagger} = c_1 + 4A/\log 2 + \int_{|x|>1} |x|/(1+|x|^2)d\Lambda(x)$, c_1 a constant and A given by Theorem 2.5. For $|t| \geq 1/2$, (4.4) is obvious.

For $|t| < 1/2$, we write

$$\begin{aligned} |\delta(t)| &\leq \int_V |tx|/(1+|tx|^2)d\eta(x) \\ &= \int_{|x|\leq 1} \frac{|tx|}{1+|tx|^2}d\eta(x) + \int_{1<|x|\leq \frac{1}{|t|}} \frac{|tx|}{1+|tx|^2}d\eta(x) + \int_{|x|>\frac{1}{|t|}} \frac{|tx|}{1+|tx|^2}d\eta(x). \end{aligned}$$

The first integral is bounded by $|t|$. By Theorem 2.5, the third one, divided by $|t|$, converges to $\int_{|x|>1} \frac{|x|}{1+|x|^2}d\Lambda(x)$ as $|t|$ tends to 0. Applying Theorem 2.5, we see that

$$\begin{aligned} \int_{1<|x|\leq \frac{1}{|t|}} \frac{|tx|}{1+|tx|^2}d\eta(x) &\leq |t| \sum_{k=0}^{\lfloor \log_2 |t| \rfloor} 2^{k+1} \eta(|x| \geq 2^k) \\ &\leq A|t| \sum_{k=0}^{\lfloor \log_2 |t| \rfloor} 2^{k+1} 2^{-k} \leq \frac{4}{\log 2} A|t| \log |t|. \end{aligned}$$

(Here by convention, when $\lfloor \log_2 |t| \rfloor$ is not an integer, the summands are for all k no larger than $\lfloor \log_2 |t| \rfloor$). Then (4.4) follows. Combining (4.4) with Lemma 4.6 we obtain

$$\lim_{t \rightarrow 0+} K_{13}(t) = 0. \quad (4.5)$$

By relations (4.2), (4.3) and (4.5), we have

$$\lim_{t \rightarrow 0+} \frac{k(tv) - 1 - i \langle v, \delta(t) \rangle}{|t|} = C_1(v).$$

□

Proof of Theorem 1.1. In view of the continuity theorem, it is enough to justify that the characteristic functions of the normalized sums S_n^x converge pointwise to a function which is continuous at zero and to show full non degeneracy of the corresponding law. The convergence follows easily from the asymptotic expansion of $k(tv)$ at $t = 0$ given by Theorem 4.2. Also if $\alpha \in [0, 2]$, using formula (1.5) for $C_\alpha(v)$, the non degeneracy proof is based on $\operatorname{Re} C_\alpha(v) < 0$ for $v \neq 0$ and is the same as in [5], since $\operatorname{supp} \Lambda$ is not contained in a hyperplane and $\Delta_v \neq 0$ is α -homogeneous. If $\alpha > 2$, the argument is the same as in [5] and is based on the order 2 differentiability of $k(t)$, since for $t \neq 0$ $r(P_{tv}) < 1$, which follows from Theorem 3.4. The invertibility of $I - z^*$ follows from the fact that $r(z^*) = r(z) < 1$, which is itself a consequence of $r(z) = \lim_{n \rightarrow \infty} (\mathbb{E}(|M|^n))^{1/n} \leq \lim_{n \rightarrow \infty} (\mathbb{E}(|M_n \cdots M_1|))^{1/n} = \kappa(1) < 1$.

□

5 On the limit laws of the normalized Birkhoff sums

Here we use the results of [22] in order to give more precise formulas for $C_\alpha(v)$ as defined by (1.5). We recall that the $\rho_\alpha(\bar{\mu})$ -stationary probability measure σ_α on \mathbb{S}^{d-1} was defined by $\Lambda = c\sigma_\alpha \otimes \ell^\alpha$ with $c > 0$. In order to write similar formulas for Δ_v ($v \in V \setminus \{0\}$) we need to distinguish two cases

I and II. In case I, $[\text{supp}\bar{\mu}]$ and $[\text{supp}\bar{\mu}]^*$ have no invariant convex cone and, using [22] we can write $\Delta_v = c^*(v)\sigma_\alpha^* \otimes \ell^\alpha$ where $c^*(v) > 0$ if $v \neq 0$ and σ_α^* is the unique $\rho_\alpha(\bar{\mu}^*)$ -stationary probability measure on \mathbb{S}^{d-1} . In case II, there are two extremal $\rho_\alpha(\bar{\mu}^*)$ -stationary measures on \mathbb{S}^{d-1} , σ'_α and σ''_α , which are symmetric of each other (hence $\sigma''_\alpha = \check{\sigma}'_\alpha$) and which are supported by the two $[\text{supp}\bar{\mu}]^*$ -minimal subsets of \mathbb{S}^{d-1} . Then, using [22], we get that there exists two nonnegative functions $c'(v)$, $c''(v)$ such that

$$\Delta_v = c'(v)(\sigma'_\alpha \otimes \ell^\alpha) + c''(v)(\sigma''_\alpha \otimes \ell^\alpha)$$

and $c'(v) + c''(v) > 0$ for $v \neq 0$.

Proposition 5.1. *With the above notations we have, if $\alpha \in [1, 2]$, $\alpha \neq 1$:*

In case I, $\Delta_v(\tilde{\Lambda}^1) = r_\alpha c^(v)$, where $r_\alpha = (\sigma_\alpha^* \otimes \ell^\alpha)(\tilde{\Lambda}^1) < 0$, $c^*(v) > 0$ if $v \neq 0$, $c^*(v)$ is α -homogeneous, and $c^*(-v) = c^*(v)$. In particular the stable limit law for S_n^x is symmetric.*

In case II, $\Delta_v(\tilde{\Lambda}^1) = c'(v)\gamma_\alpha + c'(-v)\bar{\gamma}_\alpha$, where $\gamma_\alpha = (\sigma'_\alpha \otimes \ell^\alpha)(\tilde{\Lambda}^1)$, $\text{Re}\gamma_\alpha < 0$, $c'(v) + c'(-v) > 0$ if $v \neq 0$, and $c'(v)$ is α -homogeneous.

Proof. In view of the above observations, it remains to study $c^*(v)$, σ_α^* , $c'(v)$, $c''(v)$. This follows from Proposition 2.6, in particular from the relations

$$\Delta_{tv} = t^\alpha \Delta_v \text{ for } t > 0 \text{ and } \Delta_{-v} = \check{\Delta}_v.$$

In case I, using $\Delta_v = c^*(v)(\sigma_\alpha^* \otimes \ell^\alpha)$ and the symmetry of Δ_v , Δ_{-v} , we get that σ_α^* is symmetric and $c^*(v) = c^*(-v)$. The symmetry of σ_α^* gives that $r_\alpha = (\sigma_\alpha^* \otimes \ell^\alpha)(\tilde{\Lambda}^1)$ is real and the condition $\text{Re}C_\alpha(v) < 0$ gives $r_\alpha < 0$.

In case II, the symmetry of Δ_v , Δ_{-v} gives:

$$c'(-v)(\sigma'_\alpha \otimes \ell^\alpha) + c''(-v)(\sigma''_\alpha \otimes \ell^\alpha) = c'(v)(\check{\sigma}'_\alpha \otimes \ell^\alpha) + c''(v)(\check{\sigma}''_\alpha \otimes \ell^\alpha)$$

Since σ'_α and $\sigma''_\alpha = \check{\sigma}'_\alpha$ are supported by disjoint sets, we have $c'(-v) = c''(v)$. Also since $\gamma_\alpha = (\sigma'_\alpha \otimes \ell^\alpha)(\tilde{\Lambda}^1)$, we have $(\sigma''_\alpha \otimes \ell^\alpha)(\tilde{\Lambda}^1) = (\check{\sigma}'_\alpha \otimes \ell^\alpha)(\tilde{\Lambda}^1) = \bar{\gamma}_\alpha$.

The homogeneity of $c^*(v)$, $c'(v)$ follows from the relation $\Delta_{tv} = t^\alpha \Delta_v$ if $t > 0$. \square

In order to illustrate Theorem 1.1, we consider, as in [17], the following example where $d = 2$, $\mu = p\delta_h + p'\delta_{h'}$, and $0 < p < 1$, $h = \rho \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, $h' = \left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}, b \right]$ with $\theta \notin \mathbb{Q}\pi$, $\rho > 0$, $0 < \lambda' < 1 < \lambda$, $b \neq 0$. Then $s_\infty = \infty$, $\log \kappa(s)$ is convex on $[0, \infty[$ and if ρ is sufficiently small, $L(\bar{\mu}) = \kappa'(0) < 0$. Since h' is proximal and h is an irrational similarity, condition i - p is satisfied by $[\text{supp}\bar{\mu}]$. Since $\theta \notin \mathbb{Q}\pi$, the limit set of $[\text{supp}\bar{\mu}]$ is equal to \mathbb{P}^1 and we are in case I of Proposition 5.1. If $\alpha \in [0, 2]$ with $\alpha \neq 1$, we get that the limit law of the normalized Birkhoff sum is symmetric and has Fourier transform $e^{\alpha m_\alpha c r_\alpha c^*(v)}$, where $c > 0$, $r_\alpha < 0$ and $c^*(v) = |v|^\alpha c^*(\bar{v})$ is positive for $v \neq 0$.

If $\alpha = 1$, the corresponding limit law is of Cauchy type, with Fourier transform $e^{cm_1 r_1 |v| c^*(\bar{v})}$, where $cm_1 r_1 < 0$, $c^*(\bar{v}) > 0$.

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